A geometric approach to some systems of exponential equations

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Three conjectures

Conjecture (Schanuel's conjecture)

If $z_1, \ldots, z_n \in \mathbb{C}$ *are* $\mathbb{Q}$*-linearly independent, then*

$$\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n$$

where td *stands for transcendence degree.*

Conjecture (Zilber's EAC conjecture)

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ *be an irreducible* ***free*** *and* ***rotund*** *variety. Then there is a point* $\boldsymbol{z} \in \mathbb{C}^n$ *such that* $(\boldsymbol{z}, e^{\boldsymbol{z}}) \in V$*.*

Conjecture (Zilber's quasiminimality conjecture)

$\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$ *is quasiminimal, i.e. every definable subset is countable or co-countable.*

Schanuel's conjecture

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where td *stands for transcendence degree.*

- This captures the transcendence properties of exp.
- It is out of reach. For example, it implies the algebraic independence of e and π which is a long-standing open problem. To see this, set $n = 2,\ z_1 = i\pi,\ z_2 = 1.$

Exponential equations

- Which systems of equations have solutions in $\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$?
- Consider the system $z_1 = 2z_2 + 1, e^{z_1} = 3(e^{z_2})^2$.
- exp is a homomorphism from the additive group $(\mathbb{C}; +, 0)$ to the multiplicative group $(\mathbb{C}^{\times}; \cdot, 1)$, i.e. $e^{x+y} = e^x \cdot e^y$. Therefore the above system does not have a solution.
- Let $p(X, Y) \in \mathbb{Q}[X, Y]$ be a non-zero polynomial. Does the system $e^z = 1, p(e, z) = 0$ have a solution?
- This depends on the aforementioned problem on algebraic independence of e and π. If they are independent, then the above system cannot have a solution.

Iterated exponentials

- We can get rid of iterated exponentials. For instance, given the equation $e^{e^z} = z$, we introduce new variables $z_1,\ z_2,\ w_1,\ w_2$ and consider the system $w_1 = e^{z_1},\ w_2 = e^{z_2},\ w_1 = z_2,\ w_2 = z_1$.
- This system has a solution if and only if the original equation does.
- The system has a solution iff the variety $V \subseteq \mathbb{C}^2 \times (\mathbb{C}^\times)^2$ defined by $w_1 = z_2,\ w_2 = z_1$ contains an exponential point, i.e. a point $(z_1, z_2, e^{z_1}, e^{z_2})$.
- Thus, the question is: which varieties $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ intersect the graph $\Gamma := \{(\boldsymbol{z}, \exp(\boldsymbol{z})) : \boldsymbol{z} \in \mathbb{C}^n\} \subseteq \mathbb{C}^{2n}$?

Exponential Algebraic Closedness

Conjecture (EAC)

*Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible **free** and **rotund** variety. Then there is a point $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, e^{\boldsymbol{z}}) \in V$, where $\boldsymbol{z} := (z_1, \ldots, z_n)$.*

Here and later boldface letters denote tuples.

- Let Z and W be the projections of V to $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ respectively.
- Freeness means that Z is free of additive relations and W is free of multiplicative relations.
- Rotundity is an *algebraic property* of V related to Schanuel's conjecture. For example, a rotund variety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ must have $\dim V \geq n$, and similar inequalities hold for certain projections of V.
- The *strong* EAC conjecture is about existence of generic exponential points in free and rotund varieties.

Quasiminimality

Definition

An uncountable structure is said to be *quasiminimal* if every definable subset (in one variable) is either countable or its complement is countable. Here definable can mean first-order definable, or $\mathfrak{L}_{\omega_1,\omega}$-definable or, more generally, any subset which is invariant under all automorphisms (over a countable set of parameters).

Conjecture (Zilber's quasiminimality conjecture)

$\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$ *is quasiminimal.*

- The set $2\pi i\,\mathbb{Z}$ is definable as the kernel of exp.
- In fact, $\mathbb{Z}$ is definable as the set

$$\{a \in \mathbb{C} : \forall x(\exp(x) = 1 \to \exp(ax) = 1)\}.$$

- An open question: is $\mathbb{R}$ definable in $\mathbb{C}_{\exp}$?

Zilber's pseudo-exponentiation

- Zilber constructed algebraically closed fields of characteristic 0 equipped with a unary function, called *pseudo-exponentiation*, satisfying some of the basic properties of complex exp (homomorphism, kernel), the analogues of Schanuel's conjecture and the strong EAC conjecture, and the *Countable Closure Property*.
- These can be axiomatised in the language $\mathfrak{L}_{\omega_1,\omega}(\exists^{>\aleph_0})$.
- Zilber showed that that theory is categorical in uncountable cardinals. In particular, there is a unique model $\mathbb{B}_{\exp}$ of cardinality $2^{\aleph_0}$.

Conjecture (Zilber)

$\mathbb{B}_{\exp} \cong \mathbb{C}_{\exp}$.

- This conjecture is equivalent to **Schanuel + strong EAC**.
- $\mathbb{B}_{\exp}$ is quasiminimal, so the above conjecture implies the quasiminimality conjecture.
- Bays and Kirby proved that EAC implies quasiminimality.

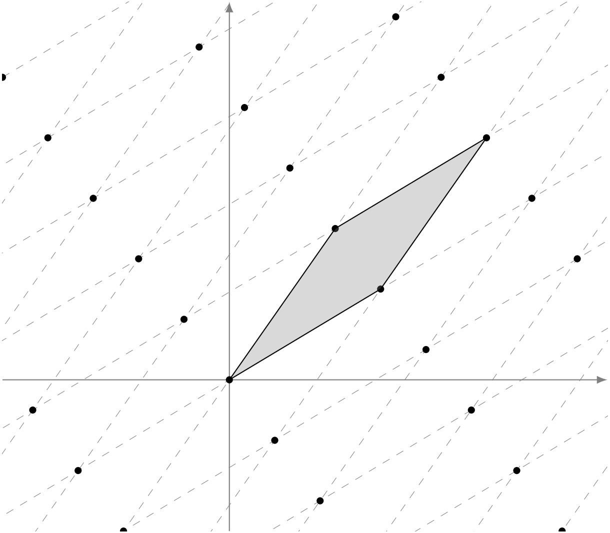
EAC: What is known

- Exponential equations in one variable can be solved. E.g. $e^z = z$ has infinitely many solutions.
- Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$, and let Z and W be the projections of V to $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ respectively.
  1. If $\dim Z = n$ (one says V has a dominant projection to $\mathbb{C}^n$), then $V \cap \Gamma \neq \emptyset$. (Brownawell-Masser, D'Aquino-Fornasiero-Terzo)
  2. EAC for $\dim Z = 1$. (Mantova-Masser)
  3. EAC for raising to generic real powers, i.e. Z is a generic real linear space and $V = Z \times W$. (Zilber)
- The analogue of (3) for abelian varieties. (Gallinaro)
- The analogue of (1) for abelian varieties. (A.-Kirby-Mantova)
- There are similar conjectures and theorems for the modular j-function. (A., Eterović, Gallinaro, Herrero, Kirby)

Elliptic curves and abelian varieties

- Let $\Lambda \subseteq \mathbb{C}$ be a lattice of rank 2, e.g. $\mathbb{Z} + i\,\mathbb{Z}$.
- The quotient $\mathbb{C}/\Lambda$ is a torus.
- It can be embedded into the projective plane $\mathbb{P}_2(\mathbb{C})$. The embedding is given by $\exp_E : z \mapsto [1 : \wp(z) : \wp'(z)/2]$ where $\wp$ is the Weierstrass $\wp$-function associated to the lattice Λ. The image is an elliptic curve.
- An elliptic curve $E \subseteq \mathbb{P}_2$ satisfies a cubic equation (in affine coordinates $y^2 = 4x^3 - g_2 x - g_3$).
- Thus, elliptic curves are connected projective algebraic groups of dimension 1.
- Abelian varieties are higher dimensional analogues of elliptic curves: connected projective algebraic groups.
- When a quotient $\mathbb{C}^g/\Lambda$ (with Λ a lattice of rank $2g$) can be embedded in a projective space, we get an abelian variety.
- The group structure of an abelian variety is commutative.
- Think of products of elliptic curves.

A lattice



EAC for abelian varieties: dominant projection

Let A be an abelian variety of dimension n (e.g. $A = E^n$) and let $\exp_A : \mathbb{C}^n \to A$ be its exponential map. Its kernel is a lattice $\Lambda \subseteq \mathbb{C}^n$ of rank $2n$.

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Theorem (A.-Kirby-Mantova)

Let $V \subseteq \mathbb{C}^n \times A$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$, that is, its projection to $\mathbb{C}^n$ has dimension n. Then there is $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$.

Moreover, we can locally parametrise all sufficiently large exponential points in V be points of Λ.

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Moreover, we can locally parametrise all sufficiently large exponential points in V be points of Λ.

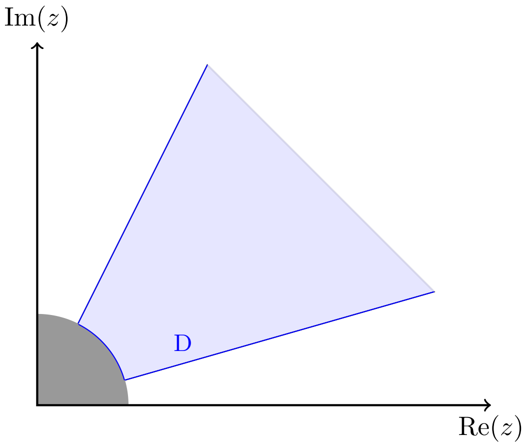
- For example, $\wp'(\wp(z)^2) = z$ has infinitely many solutions.
- Brownawell-Masser (and D'Aquino-Fornasiero-Terzo) used Newton's iterative method to approximate solutions, and in particular Kantorovich's theorem which gives criteria for these approximations to converge to an actual solution.
- Our approach is more geometric. It also works for the usual complex exponentiation. In both cases we also locally describe all sufficiently large exponential points.

Proof

- Clearly, $\dim V \geq n$. We may replace V with a subvariety and assume $\dim V = n$.
- If $(\boldsymbol{z}, \boldsymbol{w})$ are the coordinates on V then we can think of $\boldsymbol{w}$ as an algebraic expression of $\boldsymbol{z}$, for $\boldsymbol{z}$ is algebraically independent on V.
- Let $\alpha : D \to A$ be an algebraic map whose graph is contained in V.
- Here $D \subseteq \mathbb{C}^n$ is a sector domain.
- α is holomorphic on D.
- Since $(\boldsymbol{z}, \alpha(\boldsymbol{z})) \in V$, it suffices to solve the equation

$$\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z}).$$

A sector domain in $\mathbb{C}$



Mapping solutions to the lattice

- Since D is simply connected, we can choose a holomorphic branch of logarithm of α, which we denote by $G : D \to \mathbb{C}^n$.
- We can pick a fundamental domain M of Λ, and a branch of logarithm $\operatorname{Log}_A : A \to M$, and assume $G = \operatorname{Log}_A \circ \alpha$.
- In particular, since M is bounded, so is G.
- Define a map $F : D \to \mathbb{C}^n$ by

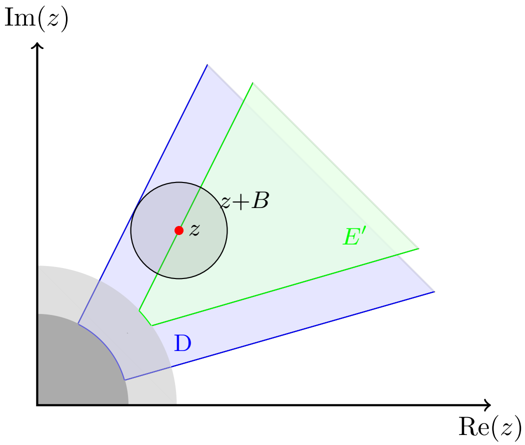
$$F : \boldsymbol{z} \mapsto \boldsymbol{z} - G(\boldsymbol{z}) = \boldsymbol{z} - \operatorname{Log}_A \alpha(\boldsymbol{z}).$$

- Clearly, $\boldsymbol{z} \in D$ solves $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ if and only if $F(\boldsymbol{z}) \in \Lambda$.
- Thus, to prove existence of solutions we need to show that the image $F(D)$ contains lattice points.

The image $F(D)$

- Since G is bounded, by Cauchy estimates, all of its first partial derivatives are bounded. In fact, shrinking D we may assume the first partial derivatives are arbitrarily small, i.e. $\|\mathrm{d}G(\boldsymbol{z})\| < \varepsilon$ on D for a small ε.
- So $\mathrm{d}F(\boldsymbol{z})$ is close to the identity matrix, hence it is non-singular.
- By the inverse function theorem, F is a local homeomorphism, hence an open map.
- So the image $E := F(D)$ is open and connected.
- Moreover, $F(\boldsymbol{z}) - \boldsymbol{z} = -G(\boldsymbol{z})$ is bounded, so F behaves like a translation near points at infinity. So E and D cannot differ by a large set.
- Let $E' := \{\boldsymbol{z} \in D : z + B \subseteq D\}$, where B is a closed ball centred at $\mathbf{0}$ containing $G(D)$. Then $E' \subseteq E$.
- E' contains a smaller sector domain, hence it contains infinitely many lattice points. Thus, $E \cap \Lambda$ is infinite, and we get infinitely many solutions to $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$.

The sets D and E' for $n = 1$



Finding all solutions in D

Lemma

F is injective on D.

Corollary

- *The map F has a holomorphic inverse $S : E \to D$.*
- *All solutions of $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ with $\boldsymbol{z} \in D$ are given by $\boldsymbol{z} = S(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in E \cap \Lambda$.*
- *Asymptotically we have $S(\boldsymbol{x}) = \boldsymbol{x} + \operatorname{Log} \alpha(\boldsymbol{x}) + o(1)$ as $|\boldsymbol{x}| \to \infty$ with $\boldsymbol{x} \in E$.*

Finding all large exponential points in V

- We patch together the maps S defined on various sector domains. This gives a multi-valued map.
- If the projection $\pi : V \to \mathbb{C}^n$ has degree d, then V is covered by the graphs of d branches of α.
- We can find all large solutions for each branch. Altogether, there will be roughly d solutions in each fundamental domain near points at infinity.

Main theorem

Theorem

Let A be a complex abelian variety of dimension n. Let $V \subseteq \mathbb{C}^n \times A$ be an irreducible subvariety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$. Then there is a subset $\Omega^* \subseteq \mathbb{P}_n$, which is open in the complex topology, such that $C := \Omega^* \setminus \mathbb{C}^n$ is Zariski open dense in $\mathbb{P}_n \setminus \mathbb{C}^n$, and there is a sheaf $\mathbf{S}$ of analytic maps on $\Omega := \Omega^* \cap \mathbb{C}^n$ taking values in $\mathbb{C}^n$ with the following properties:

1. *The image $\mathbf{S}(\Omega)$ contains Ω except possibly for a bounded strip along the boundary $\partial\Omega$.*
2. *For $\boldsymbol{\lambda} \in \Omega \cap \Lambda$, each value of $\mathbf{S}(\boldsymbol{\lambda})$ satisfies $(\mathbf{S}(\boldsymbol{\lambda}), \exp_A(\mathbf{S}(\boldsymbol{\lambda}))) \in V$. Furthermore, these are the only exponential points $(\boldsymbol{z}, \exp_A(\boldsymbol{z}))$ of V with $\boldsymbol{z}$ in Ω (except possibly near the boundary).*
3. *These exponential points are locally in d-to-1 correspondence with the points of $\Lambda \cap \Omega$.*
4. *The solutions $\mathbf{S}(\boldsymbol{\lambda})$ are asymptotically translates of the lattice.*
5. *In particular, $\{(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V\}$ is Zariski dense in V.*

The case of algebraic tori

Theorem

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a subvariety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$.
Then there is a subset $\Omega^* \subseteq \mathbb{P}_n$, which is open in the complex topology, such that $C := \Omega^* \setminus \mathbb{C}^n$ is Zariski dense and open in $\mathbb{P}_n \setminus \mathbb{C}^n$, and there is a sheaf $\mathbf{S}$ of analytic maps on $\Omega := \Omega^* \cap \mathbb{C}^n$ taking values in $\mathbb{C}^n$ with the following properties:

1. *The image $\mathbf{S}(\Omega)$ contains Ω except possibly for a narrow strip along the boundary $\partial\Omega$.*
2. *For $\boldsymbol{\lambda} \in \Omega \cap \Lambda$, each value of $\mathbf{S}(\boldsymbol{\lambda})$ satisfies $(\mathbf{S}(\boldsymbol{\lambda}), \exp(\mathbf{S}(\boldsymbol{\lambda}))) \in V$. Furthermore, these are the only exponential points $(\boldsymbol{z}, \exp(\boldsymbol{z}))$ of V with $\boldsymbol{z} \in \Omega$ (except possibly near the boundary).*
3. *These exponential points are locally in d-to-1 correspondence with the points of $\Lambda \cap \Omega$.*
4. *The solutions $\mathbf{S}(\boldsymbol{\lambda})$ are asymptotically close to lattice points.*
5. *In particular, $\{(\boldsymbol{z}, \exp(\boldsymbol{z})) \in V\}$ is Zariski dense in V.*